Matrix Powers of Column-Justified Pascal Triangles and Fibonacci Sequences

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1 Motivation

It is known that, if L_n, respectively R_n are the $n \times n$ matrices with the (i, j)th entry the binomial coefficient $\binom{i-1}{j-1}$, respectively $\binom{i-1}{n-j}$, then $L_n^2 \equiv I_n \pmod{2}$, respectively $R_n^3 \equiv I_n \pmod{2}$, where I_n is the identity matrix of dimension $n > 1$ (see for instance the problem P10735 in the May 1999 issue of American Mathematical Monthly).

The entries of L_n form a left-justified Pascal's Triangle and the entries of R_n result from taking the mirror-image of this Triangle with respect to its first column.

The questions that we ask are: can this result be extended to other primes or better yet, is it possible to find a closed form for the entries of powers of L_n and R_n?

L_n succumbs easily as we shall see in our first result. R_n in turn fights back, since closed forms for its powers are not found. However, we show a beautiful connection between matrices similar to R_n and the Fibonacci numbers. If $n = 2$, the connection is easily seen, since

$$R_2^e = \begin{pmatrix} F_{e-1} & F_e \\ F_e & F_{e+1} \end{pmatrix}.$$

1

A simple consequence of our results is that the order of L_n modulo a prime p is p, and the order of R_n modulo p divides 4 times the entry point of the Fibonacci sequence modulo p.

2 Higher Powers of L_n and R_n

The first approach that comes to mind is to find a closed form for all entries of powers of L_n, R_n. It is not difficult to obtain all the powers of L_n. Denoting the entries of the e-th power of L_n by $l_{i,j}^{(e)}$ we can prove

Theorem 1. *The entries of L_n^e are*

$$l_{i,j}^{(e)} = e^{i-j} \binom{i-1}{j-1}. \tag{1}$$

Proof. We use induction on e. The result is certainly true for $e = 1$. Now, using induction and matrix multiplication,

$$
\begin{aligned}
l_{i,j}^{(e+1)} &= \sum_{s=1}^{n} \binom{i-1}{s-1} e^{s-j} \binom{s-1}{j-1} \\
&\overset{[5],\,p.\,3}{=} \sum_{s=1}^{n} e^{s-j} \binom{i-1}{j-1} \binom{i-j}{i-s} \\
&= \binom{i-1}{j-1} \sum_{k=0}^{i-j} e^{i-j-k} \binom{i-j}{k} = \binom{i-1}{j-1}(e+1)^{i-j}.
\end{aligned}
$$

□

To prove a similar result for R_n is no easy matter. In fact, except for a few lower-dimensional cases and a few of its rows/columns, simple closed forms for the entries of R_n^e are not found.

In the sequel we consider the tableau with entries a_{ij}, $i \geq 1$, $j \geq 0$ satisfying

$$a_{i,j-1} = a_{i-1,j-1} + a_{i-1,j}, \tag{2}$$

with boundary conditions $a_{1,n} = 1$, $a_{1,j} = 0$, $j \neq n$. We shall use the following consequences of the boundary conditions and recurrence (2): $a_{i,j} = 0$ for $i + j \leq n$, and $a_{i,n+1} = 0$, $1 \leq i \leq n$ (in fact we use *only* these consequences and (2)). The matrix R will be defined as $(a_{i,j})_{i=1...n, j=1...n}$. We treat the second and third powers first, since it gives us the idea about the general case. To clear up the mysteries of some of the steps in our calculations we will refer to matrix multiplication as *m.m.* and boundary conditions as *b.c.*

Lemma 2. *The entries of the matrix R^2 satisfy*

$$b_{i,j+1} = b_{i-1,j+1} + 2b_{i-1,j} - b_{i,j}, \ 2 \leq i \leq n, \ 1 \leq j \leq n - 1 \tag{3}$$

and the entries of R^3 satisfy

$$c_{i+1,j} = 2c_{i,j} + 3c_{i,j-1} - 2c_{i+1,j-1}, \ 1 \leq i \leq n - 1, \ 2 \leq j \leq n. \tag{4}$$

Proof. Using matrix multiplication and (2) we obtain

$$
b_{i,j+1} \overset{m.m.}{=} \sum_{s=1}^{n} a_{i,s} a_{s,j+1} \overset{(2)}{=} \sum_{s=1}^{n} a_{i,s} \left(a_{s+1,j} - a_{s,j} \right) =
$$
$$
= \sum_{s=1}^{n} a_{i,s} a_{s+1,j} - \sum_{s=1}^{n} a_{i,s} a_{s,j} \overset{m.m.}{=} \sum_{s=1}^{n} a_{i,s} a_{s+1,j} - b_{i,j}. \tag{5}
$$

Therefore, denoting $S_{i,j} = \sum_{s=1}^{n} a_{i,s} a_{s+1,j}$, we obtain

$$b_{i,j+1} + b_{i,j} = S_{i,j}. \tag{6}$$

If $2 \leq i \leq n$ and $1 \leq j \leq n$,

$$
\begin{aligned}
S_{i,j} &= \sum_{s=1}^{n} \left(a_{i-1,s} + a_{i-1,s+1} \right) a_{s+1,j} \overset{t=s+1}{=} S_{i-1,j} + \sum_{t=2}^{n+1} a_{i-1,t} a_{t,j} \\
&\overset{m.m.}{=} S_{i-1,j} + b_{i-1,j} + a_{i-1,n+1} a_{n+1,j} - a_{i-1,1} a_{1,j} \overset{b.c.}{=} S_{i-1,j} + b_{i-1,j}.
\end{aligned}
$$

3

Using (6) in the previous recurrence, we obtain

$$b_{i,j+1} + b_{i,j} = b_{i-1,j+1} + b_{i-1,j} + b_{i-1,j},$$

which gives us (3).

If the relations (3) are satisfied, we obtain, for $j \geq 2$,

$$
\begin{aligned}
c_{i,j} \stackrel{m.m.}{=} \sum_{s=1}^{n} a_{i,s} b_{s,j} &\stackrel{(3)}{=} \sum_{s=1}^{n} a_{i,s}(b_{s-1,j} + 2b_{s-1,j-1} - b_{s,j-1}) \\
&= \sum_{s=1}^{n} a_{i,s} b_{s-1,j} + 2\sum_{s=1}^{n} a_{i,s} b_{s-1,j-1} - c_{i,j-1} \\
&= T_{i,j} + 2T_{i,j-1} - c_{i,j-1},
\end{aligned}
$$

where $T_{i,j} = \sum_{s=1}^{n} a_{i,s} b_{s-1,j}$. Furthermore, for $i \leq n-1$,

$$
\begin{aligned}
T_{i,j} \quad &\stackrel{(2)}{=} \quad \sum_{s=1}^{n} (a_{i+1,s-1} - a_{i,s-1}) \, b_{s-1,j} \\
&\stackrel{m.m.}{=} \quad c_{i+1,j} + a_{i+1,0} b_{0,j} - a_{i+1,n} b_{n,j} - c_{i,j} - a_{i,0} b_{0,j} + a_{i,n} b_{n,j} \\
&\stackrel{(2)}{=} \quad c_{i+1,j} - c_{i,j} + a_{i,1} b_{0,j} - a_{i,n+1} b_{n,j} \stackrel{b.c.}{=} c_{i+1,j} - c_{i,j}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
c_{i,j} \quad &= \quad T_{i,j} + 2T_{i,j-1} - c_{i,j-1} \\
&= \quad c_{i+1,j} - c_{i,j} + 2c_{i+1,j-1} - 2c_{i,j-1} - c_{i,j-1},
\end{aligned}
$$

which will produce the equations (4). $\qquad\square$

Corollary 3. *The entries of the second and third power of R can be expressed in terms of the entries of the previous row:*

$$b_{i+1,j} = b_{i,j} - \sum_{k=1}^{j-1}(-1)^k b_{i,j-k},$$
$$c_{i+1,j} = 2c_{i,j} + \sum_{k=1}^{j-1}(-1)^k 2^{k-1} c_{i,j-k}.$$

We have wondered if relations similar to (3) or (4) are true for higher powers of R. It turns out that

Theorem 4. *The entries $a_{i,j}^{(e)}$ of the e-th power of R satisfy the relation*

$$F_{e-1}a_{i,j}^{(e)} = F_e a_{i-1,j}^{(e)} + F_{e+1}a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)},$$

where F_e is the Fibonacci sequence.

Proof. We show first that the entries of R^e satisfy a relation of the form

$$\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)} \tag{7}$$

and then will proceed to find these coefficients. From Lemma 2 we observe that $\delta_1 = 0, \alpha_1 = 1, \beta_1 = 1, \gamma_1 = -1, \ \delta_2 = 1, \alpha_2 = 1, \beta_2 = 2, \gamma_2 = -1$ and $\delta_3 = 1, \alpha_3 = 2, \beta_3 = 3, \gamma_3 = -2$. Now, the coefficients of R^e satisfy, for $i, j \geq 2$,

$$
\begin{aligned}
\delta_{e-1}a_{i-1,j}^{(e)} & \overset{m.m.}{=} \sum_{s=1}^n \delta_{e-1}a_{i-1,s}a_{s,j}^{(e-1)} \\
& \overset{(7)}{=} \sum_{s=1}^n a_{i-1,s}\left(\alpha_{e-1}a_{s-1,j}^{(e-1)} + \beta_{e-1}a_{s-1,j-1}^{(e-1)} + \gamma_{e-1}a_{s,j-1}^{(e-1)}\right) \\
& \overset{m.m.}{=} \alpha_{e-1}U_{i-1,j} + \beta_{e-1}U_{i-1,j-1} + \gamma_{e-1}a_{i-1,j-1}^{(e)},
\end{aligned}
\tag{8}
$$

where $U_{i,j} = \sum_{s=1}^n a_{i,s}a_{s-1,j}^{(e-1)}$. We evaluate, for $2 \leq i \leq n$,

$$
\begin{aligned}
U_{i-1,j} & \overset{(2)}{=} \sum_{s=1}^n (a_{i,s-1} - a_{i-1,s-1})a_{s-1,j}^{(e-1)} \\
& \overset{m.m.}{=} a_{i,j}^{(e)} - a_{i-1,j}^{(e)} + a_{i,0}a_{0,j}^{(e-1)} - a_{i-1,0}a_{0,j}^{(e-1)} - a_{i,n}a_{n,j}^{(e-1)} + a_{i-1,n}a_{n,j}^{(e-1)} \\
& \overset{(2)}{=} a_{i,j}^{(e)} - a_{i-1,j}^{(e)} + a_{i-1,1}a_{0,j}^{(e-1)} - a_{i-1,n+1}a_{n,j}^{(e-1)} = a_{i,j}^{(e)} - a_{i-1,j}^{(e)},
\end{aligned}
$$

since $a_{i-1,1} = 0, i \leq n$ and $a_{i-1,n+1} = 0$. Thus,

$$\alpha_{e-1}a_{i,j}^{(e)} = (\delta_{e-1} + \alpha_{e-1})a_{i-1,j}^{(e)} + (\beta_{e-1} - \gamma_{e-1})a_{i-1,j-1}^{(e)} - \beta_{e-1}a_{i,j-1}^{(e)}.$$

5

Therefore, we obtain the following system of sequences

$$\delta_e = \alpha_{e-1}$$

$$\alpha_e = \alpha_{e-1} + \delta_{e-1}$$

$$\beta_e = \beta_{e-1} - \gamma_{e-1} \qquad (9)$$

$$\gamma_e = -\beta_{e-1}.$$

From this we deduce $\delta_e = F_{e-1}, \alpha_e = F_e, \beta_e = F_{e+1}, \gamma_e = -F_e$, where F_e is the Fibonacci

sequence, with $F_0 = 0, F_1 = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 3 can be generalized, with a little more work and anticipating (10), to obtain

the elements in the $(i+1)$th row of R^e, in terms of the elements in the previous row.

Proposition 5. *We have*

$$F_{e-1}a_{i+1,j}^{(e)} = F_e a_{i,j}^{(e)} - \sum_{k=1}^{j-1} (-1)^{k+e} \frac{F_e^{k-1}}{F_{e-1}^k} a_{i,j-k}^{(e)}.$$

3 Higher Powers of L_n and R_n modulo a prime p

As before, let L_n, respectively, R_n be defined as the matrices with entries $\binom{i-1}{j-1}$, respec-

tively, $\binom{i-1}{n-j}$. We use the notation "*matrix* $\equiv a$ (mod p)" with the meaning "*matrix* $\equiv$

aI_n (mod p)".

We ask the question of whether the order of L_n, R_n modulo a prime p is finite. We

can prove easily a result for L_n using Theorem 1.

Theorem 6. *The order of L_n $(n \geq 2)$ modulo p is p.*

Proof. We showed that the entries of L_n^e are $l_{i,j}^{(e)} = e^{i-j}\binom{i-1}{j-1}$, for any integer e. Thus, the

entries on the principal diagonal of L_n^e are all 1. If $i \neq j$, then $p\,|l_{i,j}^{(p)}$. Assume that there

is an integer e with $0 < e < p$, such that $p\,|l_{i,j}^{(e)}$, for all $i \neq j$. Take $i = 2, j = 1$. Thus,

6

$p \mid e$, a contradiction. Therefore, the integer p is the least integer $e > 0$ for which $p \mid l_{i,j}^{(e)}$, for all $i \neq j$, which proves our assertion. $\qquad\square$

We can prove the finiteness of the order of R_n modulo p in a simple manner. By the Pigeonhole Principle, there exist $s < t$, such that $R_n^s \equiv R_n^t \pmod{p}$. Since R_n is an invertible matrix ($\det R_n = (-1)^n \not\equiv 0 \pmod{p}$), $R_n^{t-s} \equiv I_n \pmod{p}$. More precise results will be proved next. In order to do that we need some known facts about the period of the Fibonacci sequence. It was shown that the period of the Fibonacci sequence modulo m (not necessarily prime) is less than or equal to $6m$ (with equality holding for infinitely many values of m) (P. Freyd, American Mathematical Monthly, December 1990, E 3410, with a solution provided in the March 1992 issue of AMM). In the case of a prime, the result can be strengthened (see the next theorem). The least integer $n \neq 0$ with the property $m \mid F_n$ is called the *entry point* modulo m.

In [1] and [7] the authors obtain (see also [6], Ch. VI-VII, for a more updated source)

Theorem 7 (Bloom-Wall). *Denote the period of the Fibonacci sequence modulo p by $\mathcal{P}(p)$. Let p be an odd prime with $p \neq 5$. If $p \equiv \pm 1 \pmod{5}$, then the period $\mathcal{P}(p) \mid (p-1)$. If $p \equiv \pm 3 \pmod{5}$, then the entry point $e \mid (p+1)$ and the period $\mathcal{P}(p) \mid 2(p+1)$.*

Remark 8. *For $p = 2$ the entry point and the period is 3. In the case $p = 5$ the entry point is 5 and the period is 20.*

Theorem 9. *If e is the entry point modulo p of F_e, then $R_{2k}^e \equiv (-1)^{(k+1)e} F_{e-1} I_{2k}$ (mod p) and $R_{2k+1}^e \equiv (-1)^{ke} I_{2k+1}$ (mod p). Moreover, $R_n^{4e} \equiv I_n$ (mod p).*

Proof. We prove by induction on e that the elements on the first row and first column of

7

R_n^e are

$$a_{1,j}^{(e)} = \binom{n-1}{j-1} F_{e-1}^{n-j} F_e^{j-1}$$

$$a_{i,1}^{(e)} = F_{e-1}^{n-i} F_e^{i-1}. \tag{10}$$

First we deal with the elements in the first row. The top equation is certainly true for $e = 1$, if we define $0^0 = 1$. Now,

$$
\begin{aligned}
a_{1,j}^{(e+1)} \;\overset{m.m.}{=}\; & \sum_{s=1}^{n} a_{1,s}^{(e)} a_{s,j} = \sum_{s=1}^{n} F_{e-1}^{n-s} F_e^{s-1} \binom{n-1}{s-1}\binom{s-1}{n-j} \\
= & \sum_{s=1}^{n} F_{e-1}^{n-s} F_e^{s-1} \binom{n-1}{j-1}\binom{j-1}{n-s} \\
= & \; F_e^{n-1}\binom{n-1}{j-1} \sum_{s=1}^{n} \left(\frac{F_{e-1}}{F_e}\right)^{n-s}\binom{j-1}{n-s} \\
= & \; F_e^{n-1}\binom{n-1}{j-1}\left(1+\frac{F_{e-1}}{F_e}\right)^{j-1} = F_e^{n-j} F_{e+1}^{j-1}\binom{n-1}{j-1}.
\end{aligned}
$$

Again by induction we prove the result for the elements in the first column. The case $e = 1$ is checked easily. Then,

$$
\begin{aligned}
a_{i,1}^{(e+1)} \;\overset{m.m.}{=}\; & \sum_{s=1}^{n} a_{i,s} a_{s,1}^{(e)} = \sum_{s=1}^{n} \binom{i-1}{n-s} F_{e-1}^{n-s} F_e^{s-1} \\
= & \; F_e^{n-1} \sum_{s=1}^{n} \left(\frac{F_{e-1}}{F_e}\right)^{n-s}\binom{i-1}{n-s} = F_e^{n-1}\left(1+\frac{F_{e-1}}{F_e}\right)^{i-1} \\
= & \; F_e^{n-i} F_{e+1}^{i-1}.
\end{aligned}
$$

Let e be the entry point modulo p of the Fibonacci sequence. By Bloom-Wall's result, $e \le p+1$. Using Theorem 4, we obtain

$$F_{e-1} a_{i,j}^{(e)} \equiv F_e a_{i-1,j}^{(e)} + F_{e+1} a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)} \pmod{p}.$$

Thus,

$$F_{e-1} a_{i,j}^{(e)} \equiv F_{e+1} a_{i-1,j-1}^{(e)} \pmod{p}. \tag{11}$$

Since $F_{e-1} + F_e = F_{e+1}$, $p \mid F_e$ and $p \nmid F_{e-1}$, we obtain $F_{e-1} \equiv F_{e+1}$ (mod p) and

$$a_{i,j}^{(e)} \equiv a_{i-1,j-1}^{(e)} \quad (\text{mod } p). \tag{12}$$

We see, from what was proved above that modulo p, the elements on the first row and column of R_n (mod p) are all zero, except for the one on the first position, which is $F_{e-1}^{n-1} \not\equiv 0$ (mod p). Using (12) we get $R_n^e \equiv F_{e-1}^{n-1} I_n$ (mod p). Using Cassini's identity $F_{e-1}F_{e+1} - F_e^2 = (-1)^e$ (see [2], p. 292), we obtain $F_{e-1}^2 \equiv F_{e+1}^2 \equiv (-1)^e$ (mod p). If $n = 2k$, then

$$F_{e-1}^{n-1} = F_{e-1}^{2k-1} \equiv \left(F_{e-1}^2\right)^k F_{e-1}^{-1} \equiv (-1)^{ke} F_{e-1}^{-1} \equiv (-1)^{(k+1)e} F_{e-1} \quad (\text{mod } p).$$

If $n = 2k + 1$, then

$$F_{e-1}^{n-1} = F_{e-1}^{2k} \equiv \left(F_{e-1}^2\right)^k \equiv (-1)^{ke} \quad (\text{mod } p).$$

The previous two congruences replaced in $R_n^e \equiv F_{e-1}^{n-1} I_n$ (mod p), will give the first two assertions of our theorem.

It is well known (a very particular case of Matijasevich's lemma) that

$$F_{2e-1} = F_{e-1}^2 + F_e^2 \equiv F_{e-1}^2 \quad (\text{mod } F_e^2),$$

so $F_{2e-1}^2 \equiv 1$ (mod p). Thus, since F_m divides F_{sm} for all m, s (in particular for $s = 2$, $m = e$), it follows that $F_{2e} \equiv 0$ (mod p) and

$$R_n^{4e} = \left(R_n^{2e}\right)^2 \equiv \left(F_{2e-1}^2\right)^{n-1} I_n \equiv I_n \quad (\text{mod } p).$$

$\square$

Remark 10. *We remark here the fact that the bound $4e$ for the order of R is tight. That can be seen by taking, for example, the prime 13, since the entry point for the Fibonacci sequence is 7, and the order of R_{4k} is 28.*

Using some Elementary Number Theory we can prove

Theorem 11. *If $p\,|\,F_{p-1}$, then $R_n^{p-1} \equiv I_n \pmod{p}$.*

Proof. We observe that, since $p\,|\,F_{p-1}$, we have $p \equiv \pm 1 \pmod 5$, otherwise $p \equiv \pm 2$ (mod 5), and by Bloom-Wall's Theorem the entry point e divides $p + 1$. Therefore, $e\,|\,p - 1$ and $e\,|\,p + 1$. Therefore, e must be 2. That is not possible, since $F_2 = 1$, which is not divisible by any prime. So, $p \equiv \pm 1 \pmod 5$ and $F_p \equiv F_{p-2} \pmod p$. Thus

$$R_n^{p-1} \equiv F_{p-2}^{n-1} I_n \equiv F_p^{n-1} I_n \pmod{p}.$$

By the previous Bloom-Wall's Theorem, $\mathcal{P}(p)\,|\,(p - 1)$, therefore $F_{p-1} \equiv 0$, $F_p \equiv 1$, $F_{p+1} \equiv 1$, etc. Thus

$$R_n^{p-1} \equiv F_p^{n-1} I_n \equiv I_n \pmod{p}.$$

$\square$

Another interesting result is the following

Theorem 12. *If $p\,|\,F_{p+1}$, then $R_{2k+1}^{p+1} \equiv I_{2k+1} \pmod{p}$ and $R_{2k}^{p+1} \equiv -I_{2k} \pmod{p}$.*

Proof. Let $p = 2$. The entry point of the Fibonacci sequence modulo 2 is $e = 3$. Since $F_2 = 1$, Theorem 9 shows the result in this case. Assume $p > 2$. We know that in this case we must have $p \equiv \pm 2 \pmod 5$. Using the known formula (see for instance Hardy and Wright [3], Theorem 180)

$$F_j = 2^{1-j}\left[\binom{j}{1} + 5\binom{j}{3} + 5^2\binom{j}{5} + \cdots\right],$$

taking $j = p$, and using Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod p$, we obtain

$$F_p \equiv 5^{(p-1)/2}\binom{p}{p} \equiv -1 \pmod{p},$$

since for the primes $\equiv \pm 2 \pmod 5$, 5 is a quadratic nonresidue.

When n is odd,

$$R_n^{p+1} \equiv F_p^{n-1} I_n \equiv (F_p^2)^{\frac{n-1}{2}} I_n \equiv I_n \pmod p.$$

Consider the case of n even. Since $F_p \equiv -1 \pmod p$, we have

$$R_n^{p+1} \equiv F_p^{n-1} I_n \equiv (-1)^{n-1} I_n \equiv -I_n \pmod p.$$

$\square$

The proofs of the previous two theorems imply

Corollary 13. *If $p \equiv \pm 1 \pmod 5$ and $p-1$ is the entry point for the Fibonacci sequence modulo p, then the period is exactly $p-1$. If $p \equiv \pm 2 \pmod 5$ and $p+1$ is the entry point for the Fibonacci sequence modulo p, then the period is exactly $2(p+1)$.*

Corollary 14. *The order of $R_n \pmod p$ is less than or equal to $2(p+1)$ and the bound is met.*

Proof. If $p \equiv \pm 1 \pmod 5$, then the order of $R_n \pmod p$ is $\leq p-1$. If $p \equiv \pm 2 \pmod 5$, then $F_p \equiv -1 \pmod p$. Therefore, $R_n^{p+1} \equiv F_p^{n-1} I_n \equiv (-1)^{n-1} I_n \pmod p$. Thus $R_n^{2p+2} \equiv I_n \pmod p$. The bound is met for all primes $p \equiv \pm 2 \pmod 5$ and all even integers n. $\square$

4   Further Problems and Results

The inverses of R_n and L_n are not difficult to find. We have

Theorem 15. *The inverse of*

$$L_n = \left(\binom{i-1}{j-1} \right)_{1 \leq i, j \leq n} \quad \text{is } L_n^{-1} = \left((-1)^{i+j} \binom{i-1}{j-1} \right)_{1 \leq i, j \leq n}.$$

11

The inverse of

$$R_n = \left(\binom{i-1}{n-j} \right)_{1 \le i,j \le n} \quad \text{is} \quad R_n^{-1} = \left((-1)^{n+i+j+1} \binom{n-i}{j-1} \right)_{1 \le i,j \le n}.$$

Proof. We have

$$\sum_s^n (-1)^{i+s} \binom{i-1}{s-1} \binom{s-1}{j-1}$$

$$\overset{[5],\, p.\,3}{=} \sum_s^n (-1)^{i+s} \binom{i-1}{j-1} \binom{i-j}{i-s}$$

$$\overset{k=i-s}{=} \binom{i-1}{j-1} \sum_{k=0}^{i-j} (-1)^k \binom{i-j}{k},$$

which is 0, unless $i = j$, in which case it is 1. A similar analysis for R_n will produce its inverse. $\qquad\square$

Another approach to find a closed form of all entries of powers of R_n would be to find all eigenvalues of R_n, and use the diagonalization of the matrix to find the entries of R_n. We found empirically and we state as a conjecture

Conjecture 16. *Denote $\phi = \frac{1+\sqrt{5}}{2}, \bar{\phi} = \frac{1-\sqrt{5}}{2}$. The eigenvalues of R_n are:*

1. $\{(-1)^{k+i}\phi^{2i-1}, (-1)^{k+i}\bar{\phi}^{2i-1}\}_{i=1,\dots,k}$, if $n = 2k$.

2. $\{(-1)^k\} \cup \{(-1)^{k+i}\phi^{2i}, (-1)^{k+i}\bar{\phi}^{2i}\}_{i=1,\dots,k}$, if $n = 2k+1$.

Another venue of research would be to study the matrices associated to other interesting sequences: Lucas, Pell, etc., and we will approach this matter elsewhere.

References

[1] D.M. Bloom, On periodicity in generalized Fibonacci sequences, American Mathematical Monthly, October 1965, pp. 856-861.

[2] R.L. Graham, D.E. Knuth, O. Patashnik, Concrete Mathematics, Adison-Wesley Publishing Company, 1989.

[3] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, Oxford-Clarendon Press, 1960.

[4] M. Petkovsek, H. Wilf, D. Zeilberger, $A = B$, A.K. Peters Ltd., Wellesley, Massachussets, 1997.

[5] J. Riordan, Combinatorial Identities, John Wiley & Sons, 1968.

[6] S. Vajda, Fibonacci & Lucas Number and the Golden Section - Theory and Applications, John Wiley & Sons, 1989.

[7] D.D. Wall, Fibonacci sequences modulo m, American Mathematical Monthly, June-July 1960, pp. 525-532.

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